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IMPORTANCE OF FIXED POINTS IN MATHEMATICS: A SURVEY

Vidyadhar V. Nalawade[1] and U. P. Dolhare[2]
[1]*Department of Mathematics, S. G. R. G. Shinde College, Paranda, Dist. Osmanabad, (M. S.), India*
[2]*Department of Mathematics, D.S.M. College, Jintur, Dist. Parbhani, (M.S.), India*

Abstract
In this research paper we have mentioned some areas where the fixed points are important. In this research paper we have explored following areas of pure and applied Mathematics where the fixed points have significant appearance.
1. Number Theory.
2. Numerical Analysis.
3. Complex Analysis.
4. Linear Algebra.
5. Transformational Geometry.
Keywords: Fixed Points, Fermat's Theorem, Mobius Transformation, Continuous Maps, Google.

I. INTRODUCTION

Fixed points appear in various branches of Mathematics and play a key role in proving some crucial results. For example the vector of PageRank values of all web pages is the fixed point of a linear transformation derived from the World Wide Web's link structure. This and such a very important results in turn are then basis of many applications in engineering, science, social sciences like economics, computers etc. This research paper is a survey of the fields where fixed points are useful.

II. FIXED POINTS

Definition 2.1 [8]. Let X be a set and $f : X \to X$ be a map from X into itself. A point x is called a "Fixed Point" of f if $f(x) = x$.

Fixed points occur at various situations in mathematics. Following are some instances where fixed points show their presence and also play very important role.

III. FIXED POINTS IN NUMBER THEORY

Fermat's little theorem in number theory is one of the celebrated results. It states as follows.

Theorem 3.1 (Fermat). If q is a prime then $k^q \equiv k(mod\ q)\ \forall$ positive integers k.

The Fermat's theorem and related results in number theory are useful in proving facts about periodic points in dynamical systems [1, 3]. Many proofs of Fermat's theorem are available in the literature. See [2, 4, 5] to see some of the proofs of this theorem. Fixed point technique can be used to establish the Fermat's little theorem. The proof of Fermat's theorem is a simple corollary of counting the fixed points of a function that we define below.

Definition 3.1 [6]. The function f from $[0,1]$ into $[0,1]$ as follows.

$$f_k(x) = \begin{cases} k.x, & if\ 0 \le x \le \dfrac{1}{k}, \\ k.x - i, & if\ \dfrac{i}{k} < x \le \dfrac{i+1}{k} \end{cases}$$

$where\ 1 \le i \le k-1\ and\ \ k \ge 2.$

Example 3.1. If $k = 6$ we get above function as,

$$f_6(x) = \begin{cases} 6x, & if\ 0 \le x \le \frac{1}{6}, \\ 6x-1, & if\ \frac{1}{6} < x \le \frac{1}{3}, \\ 6x-2, & if\ \frac{1}{3} < x \le \frac{1}{2}, \\ 6x-3, & if\ \frac{1}{2} < x \le \frac{2}{3}, \\ 6x-4, & if\ \frac{2}{3} < x \le \frac{5}{6}, \\ 6x-5, & if\ \frac{5}{6} < x \le 1. \end{cases}$$

The graph of this function is as follows.

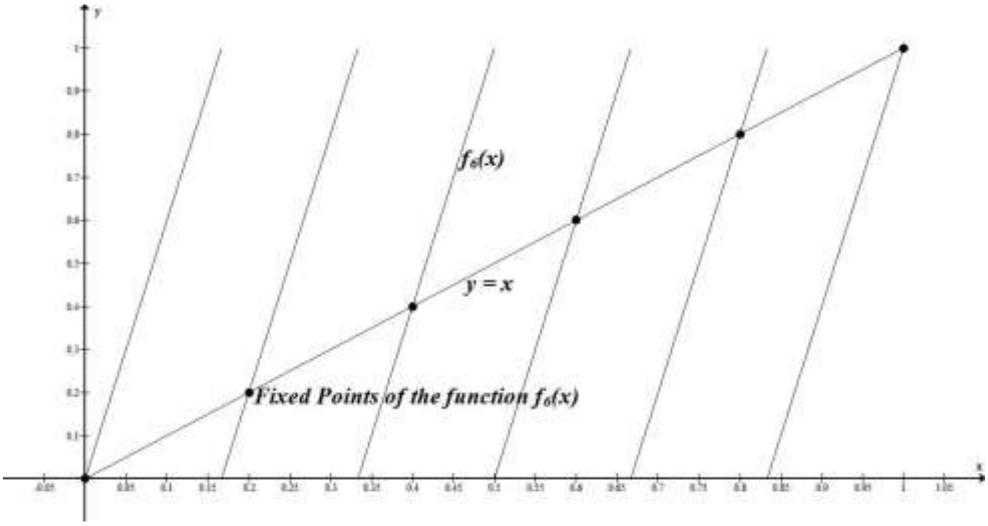


Figure 3.1. The line intersects the graph of the function $f_6(x)$ six times.

It can be easily observed that the graph of the function in the example 3.1 intersects line $x = y$ exactly 6 times, producing 6 fixed points of the function $f_6(x)$.

Lemma 3.1 [6]. The function $f_k(x)$ has exactly k number of fixed points. The function $f_k^2(x) = (f_k \circ f_k)(x)$ has exactly k^2 fixed points, the function $f_k^3(x) = (f_k \circ f_k \circ f_k)(x)$ has exactly k^3 fixed points and in general the function $f_k^n(x) = \overbrace{(f_k \circ f_k \circ f_k \circ \ldots.. \circ f_k)}^{n\,times}(x)$ has exactly k^n fixed points.

Definition 3.2 [6]. Let $f_k(x)$ be the function defined in the definition 3.1. Then "fixed point of $f_k(x)$ of period n" is a point $x \in [0,1]$ for which $f_k^n(x) = x$. That is the fixed point of $f_k(x)$ of period n is a fixed point of the function $f_k^n(x)$. Then we also call n as a "period" of x.

Definition 3.3 [6]. A "minimal period" of a fixed point $x \in [0,1]$ is the value of n such that $f_k^l(x) \ne x \, \forall l, 0 < l < n$ and $f_k^n(x) = x$.

Notation 3.1. We denote by $\mathrm{N}_n(f_k)$ the number of fixed points of minimal period n for the function f_k

Definition 3.4. For each point $x \in [0,1]$ we define "orbit of x" to be the set $\{x, f_k(x), f_k^2(x), \ldots..\}$.

Remark 3.1. If $x \in [0,1]$ has a period n, then the orbit x of contains at most n distinct elements.

Definition 3.5. If $x \in [0,1]$ has a period n, then we know that the orbit of x contains at most n distinct elements. Such orbits are called $n-$cycles.

Definition 3.6. If $x \in [0,1]$ has minimal period n, then the orbit of x contains n distinct elements: $x, f_k(x), f_k^{\,2}(x), f_k^{\,3}(x), \ldots\ldots, f_k^{\,n-1}(x)$. Such orbits are called "minimal $n-$cycles".

Lemma 3.2. The following hold.

(i) If $x_0 \in [0,1]$ is a fixed point of period n that has minimal period m, then $m|n$. That is n is divisible by m (or m divides n).

(ii) Minimal $m-$cycles are either mutually identical or disjoint. That is any two minimal $m-$cycles are either have all the elements common or no element common.

(iii) For all $n \geq 1, n|\mathrm{N}_n$ provided N_n is finite. That is N_n is divisible by n.

Lemma 3.3. The following hold.

(i) The function $f_k(x)$ defined in definition 3.1 has k^n fixed points of period n.

(ii) For all integers $k > 1$ and all integers $n \geq 1, k^n = \sum_{m|n} \mathrm{N}_m(f_k)$.

Theorem 3.2. For all integers $k \geq 2$ and all primes q, $k^q \equiv k(\bmod q)$.

Proof. We know that if n is a prime integer then the only divisors of n are 1 and n itself. By lemma 3.3, $k^q = \sum_{m|q} \mathrm{N}_m(f_k) = \mathrm{N}_1 + \mathrm{N}_q$.

Now $\mathrm{N}_1 =$ Number of fixed points of f_k of period 1=Number of fixed points of $f_k^{\,1} = f_k = k$.

Hence $k^q = k + \mathrm{N}_q$.

That is $k^q - k = \mathrm{N}_q$.

But N_q is divisible by q by lemma 3.2. Thus $k^q \equiv k(\bmod q)$.

Remark 3.2. Thus Fermat's theorem for $k \geq 2$ is a simple consequence of counting fixed points of $f_k^{\,q} = \overbrace{(f_k \circ f_k \circ f_k \circ \ldots\ldots \circ f_k)}^{q\,times}(x)$.

IV. FIXED POINTS IN NUMERICAL ANALYSIS

Finding roots of the equation of the type $f(x) = 0$ is one of the central problems in numerical analysis.

Definition 4.1. A "root" of the function $f(x)$ is a value r of x such that $f(r) = 0$. We also say this as r is a "solution" of the equation $f(x) = 0$.

Lemma 4.1. r is a root of the function $f(x)$ if and only if r is a fixed point of the function $g(x) = x - f(x)$.

Proof. Suppose r is root of the function. Then we get $f(r) = 0$.

$\therefore g(r) = r - f(r) = r - 0 = r$.

Thus r is fixed point of $g(x)$.

Conversely suppose r is a fixed point of $g(x)$.

Thus $g(r) = r$.

That is $r - f(r) = r$.

Thus $f(r) = 0$.

So r is a root of the function $f(x)$.

The lemma is thus proved.

Remark 4.1. Problem of finding a root of the function $f(x)$ thus reduces to finding a fixed point of the function $g(x) = x - f(x)$.

Example 4.1. Find the root of the function $f(x) = x^3 - 6x^2 + 11x - 6$.

Solution. Consider the function $g(x) = x - f(x) = x - \left(x^3 - 6x^2 + 11x - 6\right) = -x^3 + 6x^2 - 10x + 6.$

We observe that $x = 1, 2, 3$ are three fixed points of $g(x)$.

$g(1) = -1^3 + 6.1^2 - 10.1 + 6 = -1 + 6 - 10 + 6 = 1,$

$g(2) = -2^3 + 6.2^2 - 10.2 + 6 = -8 + 24 - 20 + 6 = 2,$

$g(3) = -3^3 + 6.3^2 - 10.3 + 6 = -27 + 54 - 30 + 6 = 3.$

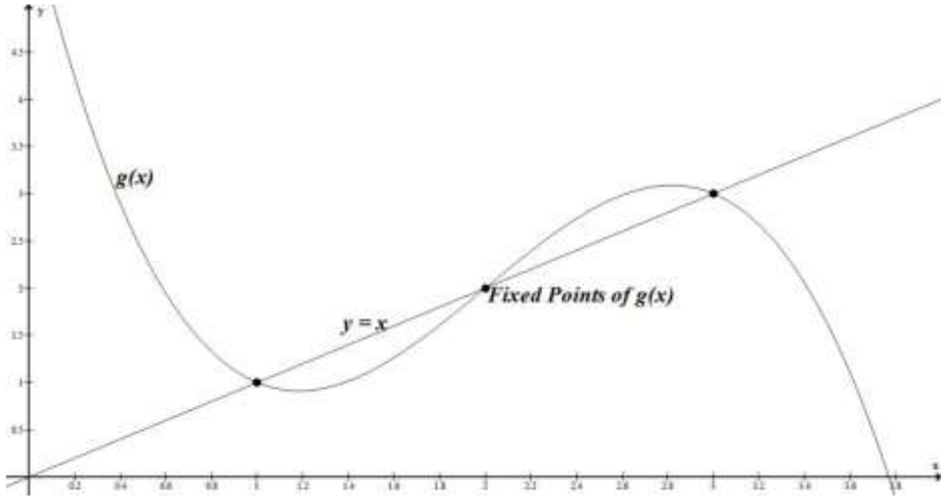


Figure 4.1. The graph of the function $g(x)$ *intersects the line* $y = x$ *thrice.*

Hence $x = 1, 2, 3$ are the roots of the equation $f(x) = x^3 - 6x^2 + 11x - 6.$

To find the fixed points of the function $g(x) = x - f(x)$ we shall use the iteration method given as under.

Step 1. Choose a point x_0 that is approximately near to the fixed point of $g(x)$.

Step 2. Compute $x_1 = g(x_0), x_2 = g(x_1), \ldots\ldots, x_n = g(x_{n-1}), \ldots\ldots$.

Step 3. If the sequence $\{x_n\}_0^\infty$ converges then it has the limit as a fixed point of $g(x)$.

V. FIXED POINTS IN COMPLEX ANALYSIS

Definition 5.1(Mobius Transformation) [7]. A transformation $M : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ defined by $M(z) = \dfrac{a_1 z + a_2}{a_3 z + a_4}, a_1, a_2, a_3, a_4 \in \mathbb{C}, a_1 a_4 - a_2 a_3 > 0$ is called a "Mobius Transformation".

Definition 5.2 (Fixed Point of Mobius Transformation) [7]. A point z_0 is called a fixed point of a Mobius transformation $M(z) = \dfrac{a_1 z + a_2}{a_3 z + a_4}$ if $M(z_0) = \dfrac{a_1 z_0 + a_2}{a_3 z_0 + a_4} = z_0$.

We shall now concentrate on the number of fixed points of a Mobius transformation.

Theorem 5.1. A Mobius transformation has at most two fixed points.

Proof. Case 1. If $a_1 = a_4 = 1, a_2 = a_3 = 0$ then the Mobius transformation becomes $M(z) = \dfrac{1.z + 0}{0.z + 1} = z$. This is an identity transformation. In this case Mobius transformation has all the points as its fixed points.

Case 2. Suppose now that none of the Mobius transformations are identity transformations. Also suppose that $z \neq 0$. We write a Mobius transformation as

$$M(z) = \frac{a_1 + \frac{a_2}{z}}{a_3 + \frac{a_4}{z}}, if\ z \neq 0.$$

Taking limit as $z \to \infty$ that is $\frac{1}{z} \to 0$ we get

$$\lim_{z\to\infty} M(z) = \frac{a_1}{a_3}.$$

If $a_3 = 0$ then we get $\lim_{z\to\infty} M(z) = \infty.$

We write this as $M(\infty) = \infty$.

Thus if $a_3 = 0$, then ∞ is a fixed point of a Mobius transformation.

Conversely suppose that ∞ is a fixed point of a Mobius transformation.

Thus $M(\infty) = \lim_{z\to\infty} \frac{a_1 + \frac{a_2}{z}}{a_3 + \frac{a_4}{z}} = \infty$.

Therefore $\frac{a_1}{a_3} = \infty.$

Thus $a_3 = 0.$

Thus a Mobius transformation has a point ∞ as a fixed point if and only if $a_3 = 0$.

Case 3. Suppose ∞ is a fixed point of a Mobius transformation. Then $a_3 = 0.$

Then the Mobius transformation becomes

$$M(z) = \frac{a_1 z + a_2}{a_4} = \frac{a_1}{a_4} z + \frac{a_2}{a_4}.$$

Letting $z = \frac{a_2}{a_4 - a_1}$ we get,

$$M\left(\frac{a_2}{a_4 - a_1}\right) = \frac{a_1}{a_4}\frac{a_2}{a_4 - a_1} + \frac{a_2}{a_4} = \frac{a_2}{a_4}\left[\frac{a_1}{a_4 - a_1} + 1\right] = \frac{a_2}{a_4}\left[\frac{a_1 + a_4 - a_1}{a_4 - a_1}\right] = \frac{a_2}{a_4}\left[\frac{a_4}{a_4 - a_1}\right] = \frac{a_2}{a_4 - a_1}.$$

Thus if ∞ is a fixed point of a Mobius transformation then $z = \frac{a_2}{a_4 - a_1}$ is also a fixed point of that Mobius transformation. We note that if $a_1 = a_4$ then that point is ∞ itself.

Case 4. Now suppose that ∞ is not a fixed point of a Mobius transformation $M(z)$. Then $a_3 \neq 0.$

Thus $\frac{a_4}{a_3} \neq \infty.$

Let z_0 be a fixed point of $M(z)$ that is $M(z_0) = z_0$.

Now $z_0 \neq -\frac{a_4}{a_3}$ because if $z_0 = -\frac{a_4}{a_3}$ then $M(z_0) = M\left(-\frac{a_4}{a_3}\right) = \frac{a_1\left(-\frac{a_4}{a_3}\right) + a_2}{a_3\left(-\frac{a_4}{a_3}\right) + a_4} = \infty.$

Thus $z_0 = \infty$ and then $a_3 = 0$. This is a contradiction.

Consider $a_3 z_0 + a_4$, which is not 0 because $z_0 \neq -\frac{a_4}{a_3}$.

Multiply by $a_3 z_0 + a_4$ to $M(z_0) = \frac{a_1 z_0 + a_2}{a_3 z_0 + a_4} = z_0$ we get,

$a_1 z_0 + a_2 = z_0 (a_3 z_0 + a_4)$

$\therefore a_1 z_0 + a_2 = a_3 z_0^{\ 2} + a_4 z_0$

$\therefore a_3 z_0^{\ 2} + (a_4 - a_1) z_0 - a_2 = 0$

The last equation is quadratic equation in z_0.

This equation has either

(1) Two real solutions or

(2) One real solution or

(3) Two complex conjugate solutions.

These solutions are the fixed points of $M(z)$.

Remark 5.1: - Thus a non-identity Mobius transformation has at most two fixed points. Any Mobius transformation which has three or more fixed points then the Mobius transformation is identity transformation. Then the Mobius transformation has all the fixed points.

VI. FIXED POINTS IN LINEAR ALGEBRA

The role of fixed points in linear algebra and its' applications is illustrated in the following diagram.

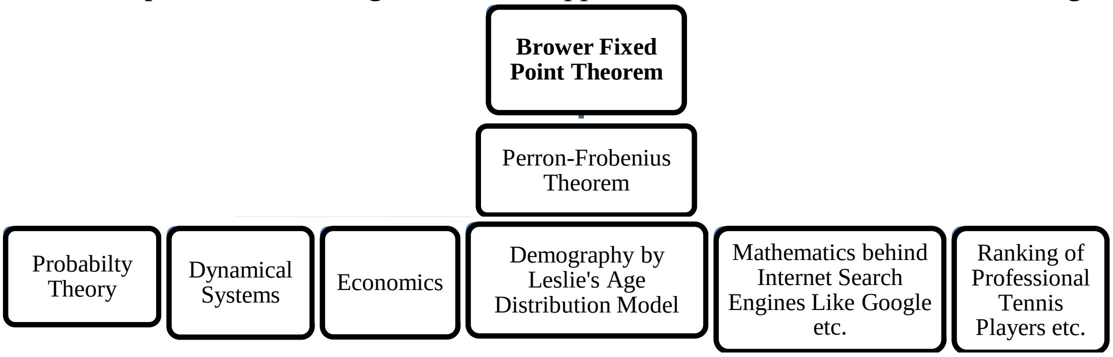


Figure 6.1. Brower's Fixed Point Theorem is basis for the proof of Perron-Frobenius theorem which has many important applications.

Definition 6.1 (Closed Unit Ball in $\mathbb{R}^n$). A Closed Unit Ball B^n in $\mathbb{R}^n$ is defined as the set

$$\mathrm{B}^n = \left\{ u = (u_{1,} u_2, u_3, \ldots.., u_n) / u_i \in \mathbb{R}, 1 \leq i \leq n, \|u\| = \left(u_1^{\ 2} + u_2^{\ 2} + u_3^{\ 2} + \ldots.. + u_n^{\ 2}\right)^{\frac{1}{2}} \leq 1 \right\}$$

Theorem 6.1 (Brower Fixed Point Theorem) [8]. Any continuous map f from the closed unit ball B^n in $\mathbb{R}^n$ to itself has a fixed point.

Before we actually state and prove the Perron-Frobenius theorem in linear algebra, by using Brower fixed point theorem, we define the followings.

Definition 6.2. A $n \times n$ matrix $A = \left(a_{ij}\right)$ is said to be non-negative (respectively positive) and we write $A \geq 0$ (respectively $A > 0$) if $a_{ij} \geq 0$ (respectively $a_{ij} > 0$) for all i, j.

Definition 6.3. A vector X in $\mathbb{R}^n$ is called a non-negative (respectively positive) and we write $X \geq 0$ (respectively $X > 0$) if X is a non-negative (respectively positive) when regarded as a matrix.

Definition 6.4. The set of all eigenvalues of a matrix is called its "spectrum".

Definition 6.5. The largest value of the modulus of an eigenvalue of a matrix A is called the "spectrum radius" of A. It is denoted by $r(A)$.

Lemma 6.1. If $A > 0$ and u is an eigenvector of of A with $u \geq 0$, then $u > 0$.
Now we state and prove Perron-Frobenius theorem.
Theorem 6.2 (Perron-Frobenius Theorem). Let $A = (a_{ij})$ be a real strictly positive $n \times n$ matrix. Then
(1) A has a positive eigenvalue λ with $r(A) = \lambda$.
(2) λ is unique such an eigenvalue.
(3) The corresponding eigenvector is also strictly positive.
(4) A has no other non-negative eigenvector. That is all the other eigenvectors are not non-negative.
Proof of (1) and (2). Let S_n be the unit sphere with the centre origin in $\mathbb{R}^n$.
Define $S = \left\{ u = (u_1, u_2, \ldots.., u_n) / \|u\| = 1, u_i \geq 0, \text{ for all } i = 1, 2, 3, \ldots.., n \right\}$.
It is easy to check that S is homeomorphic to the closed unit ball B^{n-1} in $\mathbb{R}^{n-1}$.
Define a function $f : S \to S$ by $f(u) = \dfrac{Au}{\|Au\|}$. This function is clearly continuous. So we can apply Brower's fixed point theorem on f. ***By applying Brower's fixed point theorem on*** f ***we get a fixed point of*** f***, say*** $u_0 = (u_{0,1}, u_{0,2}, u_{0,3}, \ldots.., u_{0,n})$. Thus we have

$$f(u_0) = \frac{Au_0}{\|Au_0\|} = u_0 .$$

Let $\lambda = \|Au_0\|$. So that we get $Au_0 = \lambda u_0$. Thus λ is a eigenvalue of the matrix A.
Clearly $\lambda > 0$.
Proof of (3). By the definition of the set S, all the components of u_0 are non-negative and $A > 0$. Thus $Au_0 > 0$ (see lemma 6.1) and hence the eigenvector $u_0 > 0$.
Proof of (4). Next we shall show that λ has no other eigenvector. This we show by contradiction method. Suppose that there is another eigenvector $u_0' = (u_{0,1}', u_{0,2}', u_{0,3}', \ldots.., u_{0,n}')$, independent of u_0. As we know $A > 0$ and $\lambda > 0$, by the similar argument as above we get $u_0' > 0$.
Let $t = \min_i \dfrac{u_{0,i}}{u_{0,i}'} = \dfrac{u_{0,k}}{u_{0,k}'}$ for some k. Consider the vector $u_0'' = u_0 - tu_0'$.
Then we have $Au_0'' = A(u_0 - tu_0') = Au_0 - tAu_0' = \lambda u_0 - t\lambda u_0' = \lambda(u_0 - tu_0') = \lambda u_0''$. Therefore u_0'' is also an eigenvector of A corresponding to the eigenvalue λ. But $u_k'' = u_{0,k} - tu_{0,k}' = u_{0,k} - \dfrac{u_{0,k}}{u_{0,k}'} u_{0,k}' = 0$.
This is a contradiction to the fact that u_0'' is an eigenvector of A with the eigenvalue λ since u_0 and u_0' are linear independent. Thus A has no other non-negative eigenvector.
Definition 6.6 [8]. A $n \times n$ matrix is called "Markov matrix"/"Stochastic matrix" if it is non-negative and the sum of the elements of each column is 1.
Theorem 6.3 [8]. A Markov matrix has always an eigenvalue 1 and all other eigenvalues are in absolute value smaller than or equal to 1.
Perron-Frobenius theorem and theorem 6.3 are become important in application in computers and various applications as illustrated in figure 6.1. We shall consider one of the many important applications shown in figure 6.1.
Example 6.1. Application of Brower's fixed point theorem to Google.
Twenty first century is the era of information and technology. Large information is available on a single click on computers. Also smart phones enable us to browse any information at our finger tips. However for a layman it is question of wonder that how a given query is answered on a computer with in a fraction of seconds. Today search engines (like Google, Yahoo) are like a huge library. Massive information is stored in different formats like PDF, DJVU etc. Anybody can add any

information at any time in any format to this library. And the important thing is that there is no management and no librarian to monitor the transactions. Its'wonderful that under such situations one can find a particular document in few seconds.

Google works as follows:

1. Explore the web and locate the web pages which have public access.
2. Then the engine index the data found in above step.
3. On the probability basis the engine rate the importance of each page in the data base, so that more important pages are appear on the screen of a computer.

We explain in detail the last step 3. Google uses the system PageRank to rank the importance of a webpage. This system is based on the following ideas.

- On internet a page is linked to another page. For example a Facebook page has a link that takes us to a Twitter page. This we can imagine as if one page votes for another page.
- A web page is ranked higher than another if there are more links to it. That is a page is ranked higher if more pages votes for it.
- Further one vote becomes more important than the other if it is obtained from more important page.

To understand the above concept of calculating the rating of a web page we take one example. Let the webpage P has pages V1, V2, V3, ….., Vn as its' voter pages. Let #P be the number of votes given by page P to other webpages.

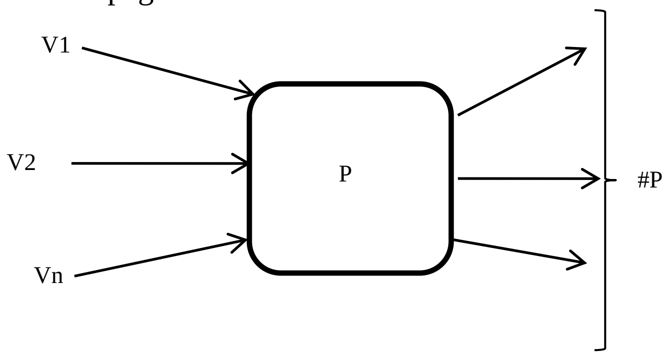


Figure 6.2. Votes casted to and casted by the webpage P

Further suppose the webpage Vj casts n_j votes to other pages and out of j votes only one is casted for the page P. Then it increases the importance of the page by PR(Vj)/n_j. Here PR(Vj) is the PageRank of the page Vj. Therefore assume that the internet has m pages P1, P2, P3, ….., Pm. Let Nk be the number of outgoing links from Pk. Then

$$PR(Pk) = \sum_{j \neq k} \frac{PR(Pj)}{Nj}, k = 1, 2, 3, ...m. \qquad (1)$$

We also note that

(1) Different PageRank form a discrete probability distribution (or probability density function) over the web pages. Hence Sum of all PageRanks = 1,

(2) PR(Pk) corresponds to the principal eigenvector of the normalized link matrix of the web. Therefore it can be calculated from above algorithm.

Example 6.2. Suppose a web page contains four pages P1, P2, P3, P4. The links from one page to other are given in the following figure.

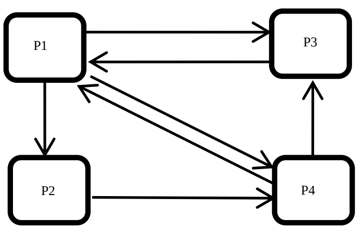


Figure 6.3. Arrows shows the links between the web pages

Let PR(P1) = x, PR(P2) = y, PR(P3) = z, PR(P4) = w. Then we get
x = z/1+w/2, y = x/3, z = x/3+y/2+w/2, w = x/3+y/1.
Therefore if we consider the vector

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

We have the system $AX = X$, with

$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Here A has eigenvalue 1 and its unique corresponding eigenvector from Perron-Frobenius theorem 6.2 (which we have proved by using ***Brower's fixed point theorem***). This is

$$X = \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}.$$

Normalizing in order to have stochastic vector we get

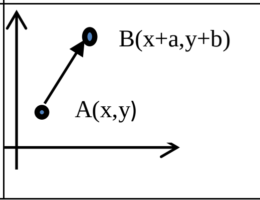
$$X = \frac{1}{31}\begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}.$$

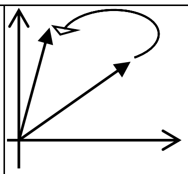
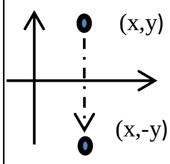
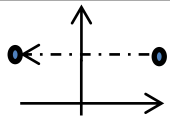
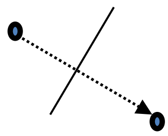
This ranking makes the page P1 the most important web page.

VII. FIXED POINTS IN TRANSFORMATIONAL GEOMETRY

Definition 7.1 (Isometry). "Isometry" is a distance and angle preserving transformation. So an isometry also preserves shapes.
Definition 7.2 (Fixed Point of an isometry). The fixed points of an isometry are those points in the plane whose images are themselves.
Remark 7.1. The number of **fixed points** of isometries helps us to **characterize** all the isometries.
Various types of isometries and their characterization by the number of fixed points is shown by the following table.

Table 7.1. Chart showing the number of fixed points of an isometry geometrical transformation

Sr. No.	Isometry	Definition	Figure	Number of fixed points
1	Translation	Under translation a point $A(x, y)$ is translated to the point $B(x+a, y+b)$		No fixed points or all the points are fixed if $a = 0, b = 0$ In this case translation becomes an identity translation

2	Rotation	Under rotation about an angle θ a point $A(x, y)$ is translated to the point $B(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$		Roration has unique fixed point that is its point of rotation.
3	Reflection	Under the reflection over X axis a point $A(x, y)$ is translated to the point $B(x, -y)$. And under the reflection over Y axis point $A(x, y)$ is translated to the point $B(-x, y)$.	  	All the points on the line of reflection are fixed.

VIII. CONCLUSION

From this research article it is now crystal clear that the fixed points are very useful in Mathematics. Analysis of mappings and their fixed points enable us to successfully handle many situations in pure and applied Mathematics. Main braches of Pure Mathematics like Real Analysis, Complex Analysis, Number Theory and branches of Applied Mathematics like Linear Algebra, Numerical Analysis utilise fixed points and related theorems. So it is essential to explore more on fixed points. In the coming research paper we shall further explore the areas like economics, game theory etc.

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